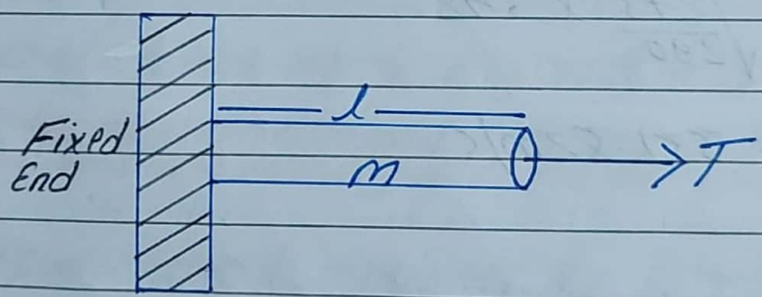
WAVES IN PIPES AND STRINGS

MUSICAL INSTRUMENTS

Membrane	String	Pipes
Eg: Dholak	Eg: Guitar	Eg Whistle
Table	Sitar	Flute
Drum	Ek tare	Clarinet
Madal	Sarangi	Trumpet

Velocity of transverse wave along a stretched string.



Experimentally it has been found that the velocity of transverse wave along a stretched string depends upon:-

i) Tension (T) in the string

i.e $v \propto T^a$ — (1)

ii) Mass (m) of the string

i.e $v \propto m^b$ — (2)

iii) length (l) of the string

i.e $\nu \propto l^c$ — (3)

Combining relations (1), (2) and (3), we get

$$\nu \propto T^a m^b l^c$$

$$\Rightarrow \nu = kT^a m^b l^c \quad \text{— (4)}$$

Where k is a constant of proportionality. (k is dimensionless constant).

Writing the dimension of corresponding physical quantities in (4) we get

$$[LT^{-1}] = [MLT^{-2}]^a [M]^b [L]^c$$

$$\Rightarrow [LT^{-1}] = [M^a L^a T^{-2a}][M^b][L^c]$$

$$\Rightarrow [M^0 L^1 T^{-1}] = [M^{a+b} L^{a+c} T^{-2a}]$$

Equating the powers of corresponding physical quantities, we get

$-2a = -1$	$a + b = 0$	$a + c = 1$
$2a = 1$	$\frac{1}{2} + b = 0$	$\frac{1}{2} + c = 1$
$a = \frac{1}{2}$	$b = -\frac{1}{2}$	$c = 1 - \frac{1}{2} = \frac{1}{2}$

Substituting the values of a, b and c in 4 we get

$$\nu = k \cdot T^{1/2} \cdot m^{-1/2} \cdot l^{1/2}$$

$$\Rightarrow \nu = k \cdot T^{1/2} \cdot \frac{1}{m^{1/2}} \cdot l^{1/2}$$

$$\Rightarrow \nu = k\left(\frac{Tl}{m}\right)^{1/2}$$

$$\Rightarrow \nu = k\sqrt{\frac{T.l}{m}}$$

$$\Rightarrow \nu = k\sqrt{\frac{T}{\left(\frac{m}{l}\right)}}$$

$$\Rightarrow \nu = k\sqrt{\frac{T}{\mu}} \quad \left[\begin{array}{l} \text{'}l\text{'}\ m = m \\ \therefore 1m = \frac{m}{l} \end{array}\right]$$

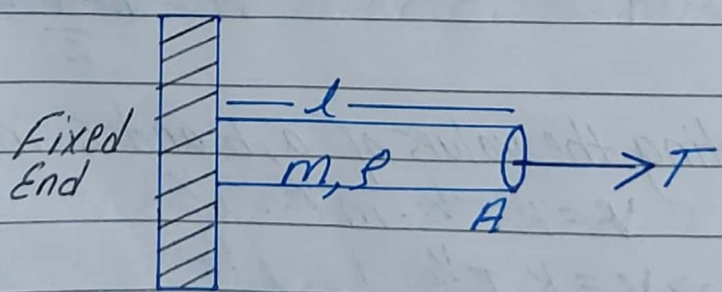
Hence, $\frac{m}{l} = \mu$, the mass per length of the string

By numerical analysis, $k=1$

$$\therefore \boxed{\therefore \nu = \sqrt{\frac{T}{\mu}}}$$

This gives the velocity of transverse wave along a stretched string.

Velocity of transverse wave along a stretched string (in terms of cross-sectional area/diameter of the string).



The velocity of transverse wave along a stretched string is given by,

$$v = \sqrt{\frac{T}{\mu}}$$

$$\Rightarrow v = \sqrt{\frac{T}{\left(\frac{m}{l}\right)}} \qquad [\because \mu = m/l]$$

$$\Rightarrow v = \sqrt{\frac{Tl}{m}}$$

$$\Rightarrow v = \sqrt{\frac{T.l}{\rho.V}} \qquad [\because \rho = m/V)$$

$$\Rightarrow v = \sqrt{\frac{T.l}{\rho \times A \times l}}$$

$$\therefore \boxed{\therefore v = \sqrt{\frac{T}{\rho A}}}$$

This gives the velocity of transverse wave along a stretched string in terms of cross-sectional area of the string.

Again,

$A = \frac{\pi d^2}{4}$ where d is the diameter of string.

$$\therefore v = \sqrt{\frac{T}{\rho \times \frac{\pi d^2}{4}}}$$

$$\Rightarrow v = \sqrt{\frac{4T}{\rho \pi d^2}}$$

$$\boxed{\nu = \frac{2}{d}\sqrt{\frac{T}{\pi\rho}}}$$

This gives the velocity of transverse wave along a stretched strings in terms of diameter of the string.

From the above equations,

$$\boxed{\nu \propto \frac{1}{\sqrt{A}}} \quad \text{Or,} \quad \boxed{\nu \propto \frac{1}{d}}$$

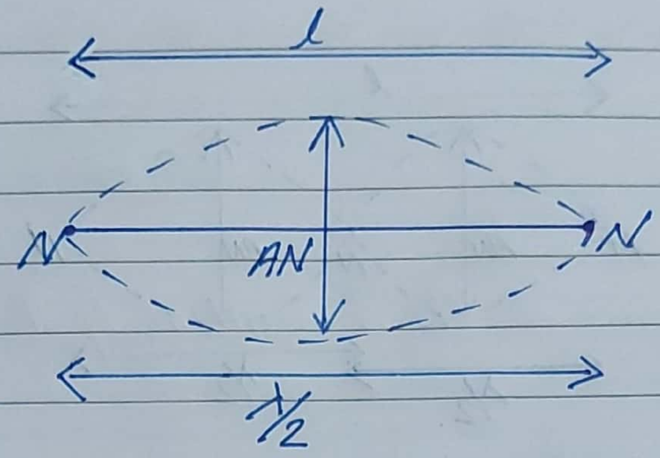
Thus, the transverse wave travels faster in a thinner string than in a thicker string.

Normal modes of vibration in a stretched string

Consider a stretched string attached to both fixed ends when the string is plucked from the middle and left itself, the transverse wave travel along the string. The wave generated is then reflected from the fixed end and they superpose forming the stationary transverse wave.

The formation of waves in the string can be studied for following modes of vibration.

i. Fundamental mode of vibration



In this case,

$$l = \lambda/2$$

$$\Rightarrow \lambda = 2l$$

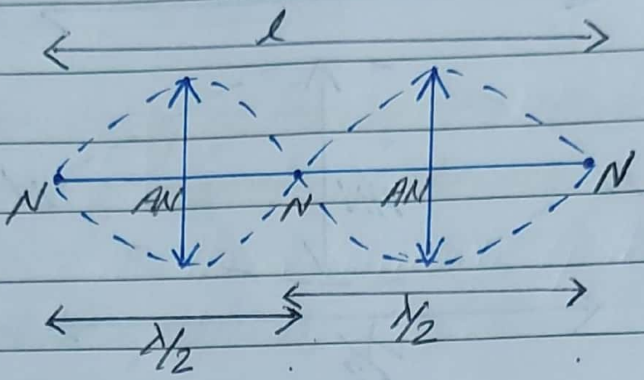
If 'v' is the velocity of transverse wave along a stretched string, then the frequency of vibration of the string is,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow \boxed{f_1 = \frac{v}{2l}}$$

This gives the fundamental frequency of vibration of the string. It is the lowest frequency produced by the vibrating string. This frequency is the first harmonic of the string.

ii. First overtone



In this case,

$$l = \frac{\lambda}{2} + \frac{\lambda}{2}$$

$$\Rightarrow l = \frac{2\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{2}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{2}\right)}$$

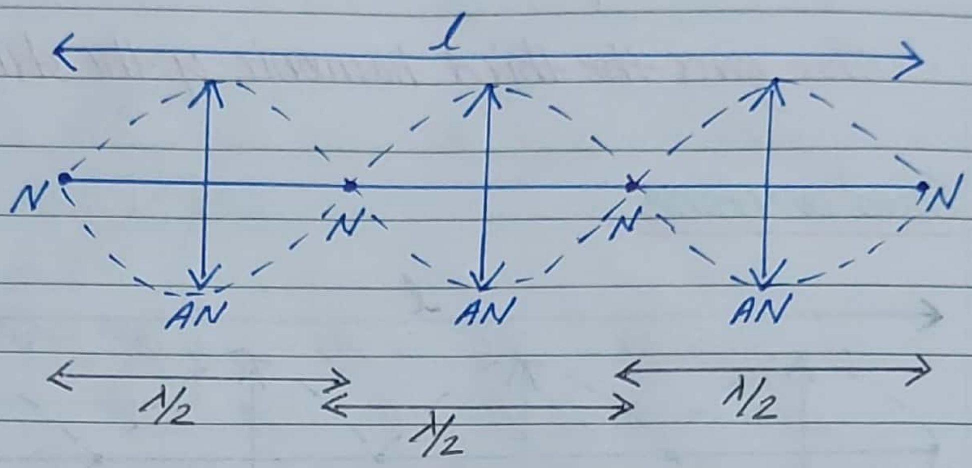
$$\Rightarrow f = \frac{2v}{2l}$$

$$\Rightarrow f_2 = 2\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_2 = 2f_1}$$

∴ This gives the second harmonic of the string.

iii. Second overtone



In this case,

$$l = \frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2}$$

$$\Rightarrow l = \frac{3\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{3}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{3}\right)}$$

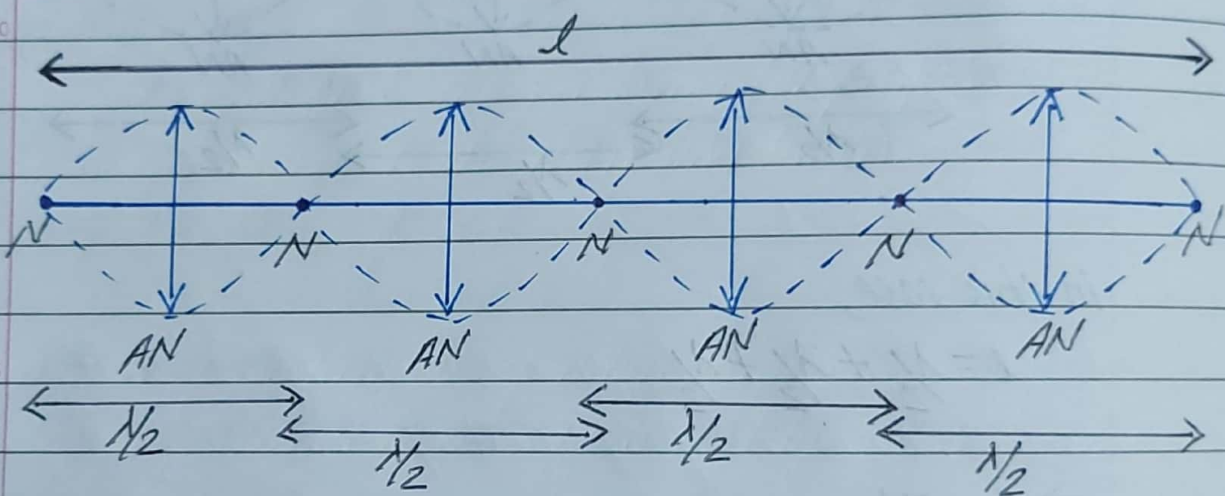
$$\Rightarrow f = \frac{3v}{2l}$$

$$\Rightarrow f = 3\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_3 = 3f_1}$$

$\therefore$ This gives the third harmonic of the string.

iv. Third overtone



In this case,

$$l = \frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2}$$

$$\Rightarrow l = \frac{4\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{4}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{4}\right)}$$

$$\Rightarrow f = \frac{4v}{2l}$$

$$\Rightarrow f = 4\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_4 = 4f_1}$$

$\therefore$ This gives the fourth harmonic of the string.

Conclusion

i. n^{th} overtone $\Rightarrow (n+1)^{th}$ harmonic

$$\boxed{f_{n+1} = (n+1)f_1}$$

This gives the $(n+1)^{th}$ harmonic of the stretched string.

Laws of vibration of stretched string

The frequency of fundamental made of a transverse wave in a stretched string is given by,

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$$f = \frac{\nu}{2l}$$

or, $$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

where T is the tension in the stretched string and μ, the mass per unit length and l is resonating length.

From this expression, it follows that there are three laws of transverse vibration of stretched string:-

i. The law of length:- The fundamental frequency is inversely proportional to the resonating length of the string.

i.e. $\boxed{f \propto \frac{1}{l}}$

ii. The law of tension:- The fundamental frequency is directly proportional to the square root of the stretching force or tension in the string.

i.e. $\boxed{f \propto \frac{1}{\sqrt{T}}}$

iii. The law of mass:- The fundamental frequency is inversely proportional to the square root of the mass per unit length of the string.

i.e. $\boxed{f \propto \frac{1}{\sqrt{\mu}}}$

Organ pipes and its type

A hollow wooden or metallic tube used to produce musical sound is called an organ pipe. It is a wind instrument such as a flute, whistle, clarinet etc.

Types of organ pipe

i. Closed organ pipes

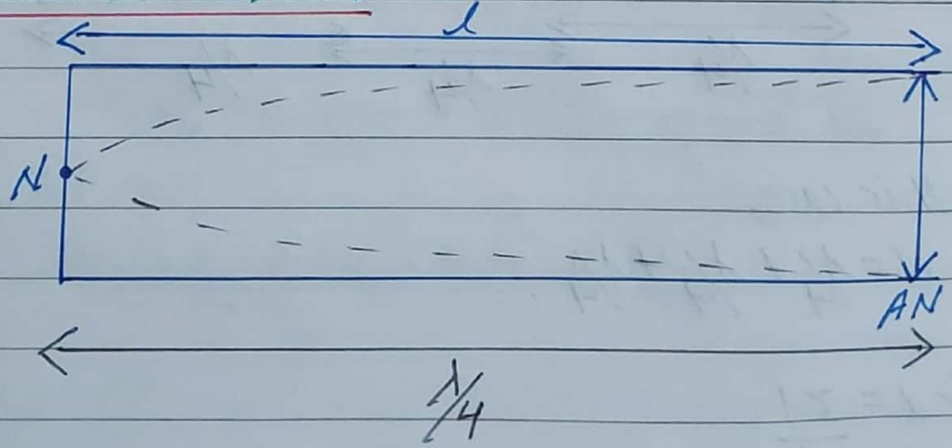
Such type of organ pipes which are closed at one end and open at another end are called as closed organ pipes.

Eg: whistle, pen lid, etc.

Normal modes of vibration in a closed organ pipe

When a blast of air is blown into a closed organ pipe, at the open end, a wave thus travels through the pipe and it is reflected at the closed end. Due to superposition of incident and reflected waves, longitudinal stationary waves are formed.

i. Fundamental mode :-



In this case,

$$l = \lambda/4$$

$$\Rightarrow \lambda = 4l$$

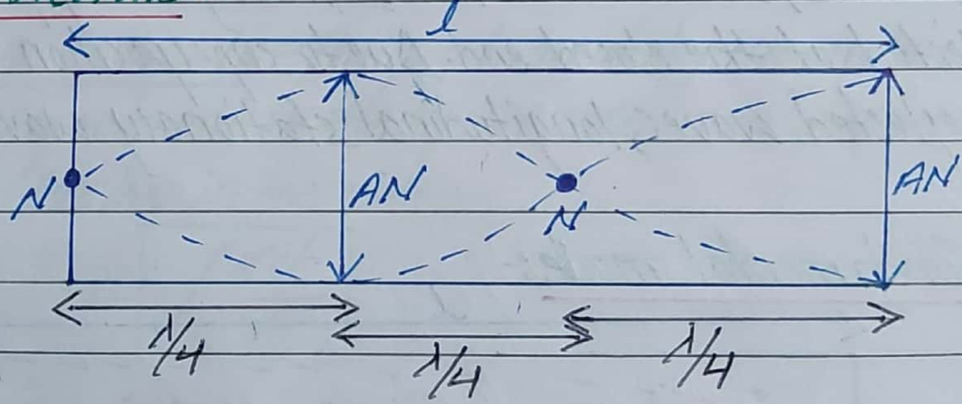
Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow \boxed{f_1 = \frac{v}{4l}}$$

This gives the fundamental frequency of the closed organ pipe. It is the lowest frequency produced by the pipe. This frequency is the first harmonic of the pipe.

ii. First overtone :-



In this case,

$$l = \lambda/4 + \lambda/4 + \lambda/4$$

$$\Rightarrow l = \frac{3\lambda}{4}$$

$$\Rightarrow \lambda = \frac{4l}{3}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{4l}{3}\right)}$$

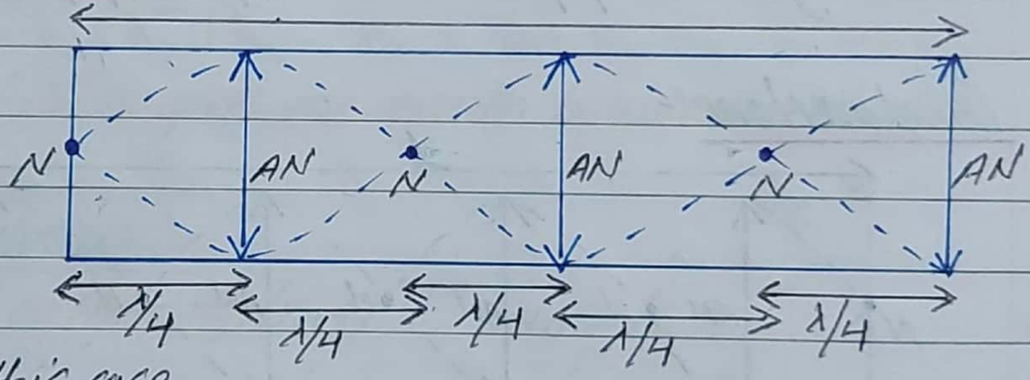
$$\Rightarrow f = \frac{3v}{4l}$$

$$\Rightarrow f = 3\left(\frac{v}{4l}\right)$$

$$\boxed{\therefore f_2 = 3f_1}$$

$\therefore$ This gives the third harmonic of the pipe.

iii) Second overtone :-



In this case,

$$l = \frac{\lambda}{4} + \frac{\lambda}{4} + \frac{\lambda}{4} + \frac{\lambda}{4} + \frac{\lambda}{4}$$

$$\Rightarrow l = \frac{5\lambda}{4}$$

$$\lambda = \frac{4l}{5}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{4l}{5}\right)}$$

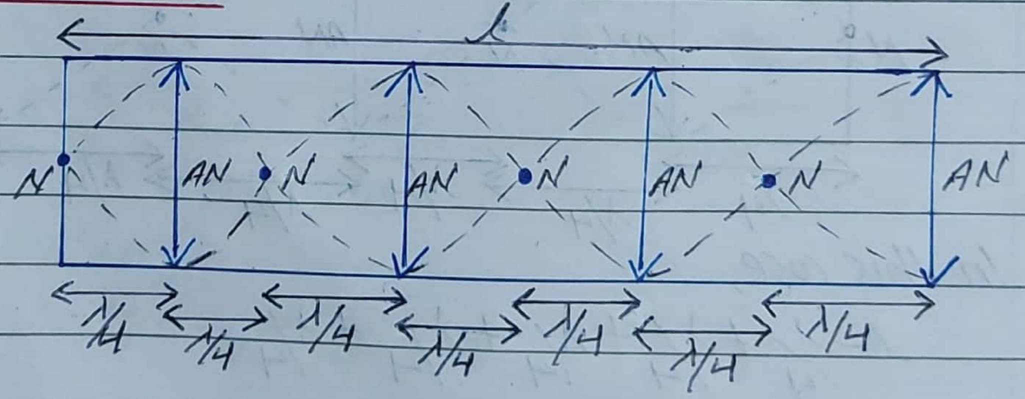
$$\Rightarrow f = \frac{5v}{4l}$$

$$\Rightarrow f = 5\left(\frac{v}{4l}\right)$$

$$\Rightarrow \boxed{f_3 = 5f_1}$$

$\therefore$ This gives the fifth harmonic of the pipe.

iv. Third overtone:-



In this case,

$$l = \lambda/4 + \lambda/4 + \lambda/4 + \lambda/4 + \lambda/4 + \lambda/4 + \lambda/4$$

$$\Rightarrow l = \frac{7\lambda}{4}$$

$$\Rightarrow \lambda = \frac{4l}{7}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{4l}{7}\right)}$$

$$\Rightarrow f = \frac{7v}{4l}$$

$$\Rightarrow f = 7\left(\frac{v}{4l}\right)$$

$$\Rightarrow \boxed{f_4 = 7f_1}$$

$\therefore$ This gives the seventh harmonic of the pipe.

Conclusion

(i) n^{th} overtone $\Rightarrow (2n+1)^{th}$ harmonic

$$\boxed{\therefore f_{(2n+1)} = (2n+1)f_1}$$

This gives the $(2n+1)^{th}$ harmonic of the closed organ pipe.

(ii) Only odd harmonics are present. So, the quality of sound is poor. Hence, closed organ pipes cannot be used as musical instuments.

Open organ pipes

Such type of organ pipes which are open at both the ends are called open organ pipes.

Examples:- Flute, clarinet, trumpet etc.

Normal modes of vibration in an open organ pipe

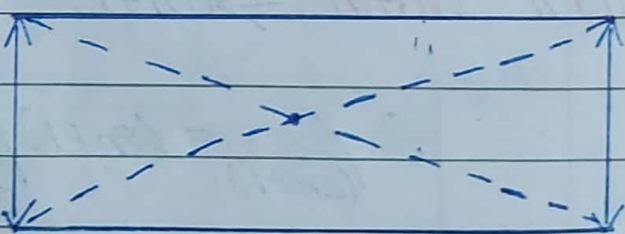
In an open organ pipe, there is the formation of stationary longitudinal waves.

i. Fundamental mode:-

When air is blown into the pipe through one end, a wave travels through the tube to the next end from where it is reflected. Due to superposition of the incident and reflected waves, a stationary longitudinal wave is set up in air in the pipe.

In this case,

$$l = \lambda/2$$



$$\Rightarrow \lambda = 2l$$

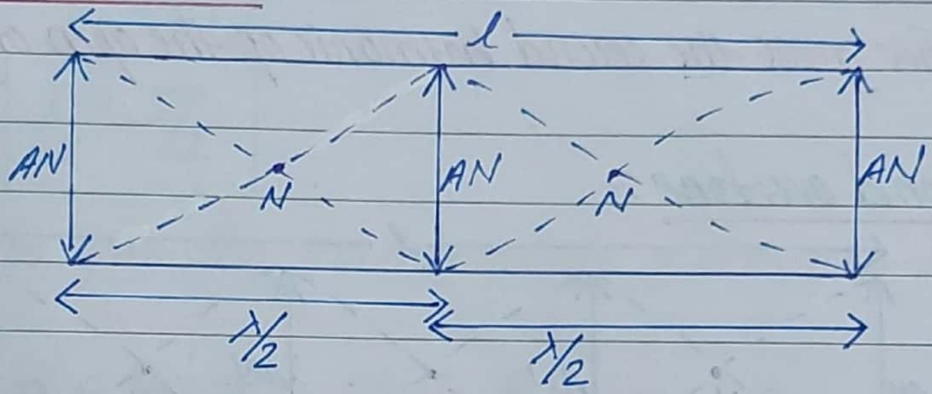
Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow \boxed{f_1 = \frac{v}{2l}}$$

This gives the fundamental frequency of vibration of the open organ pipe.

ii. First overtone :-



In this case,

$$l = \lambda/2 + \lambda/2$$

$$\Rightarrow l = \frac{2\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{2}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{2}\right)}$$

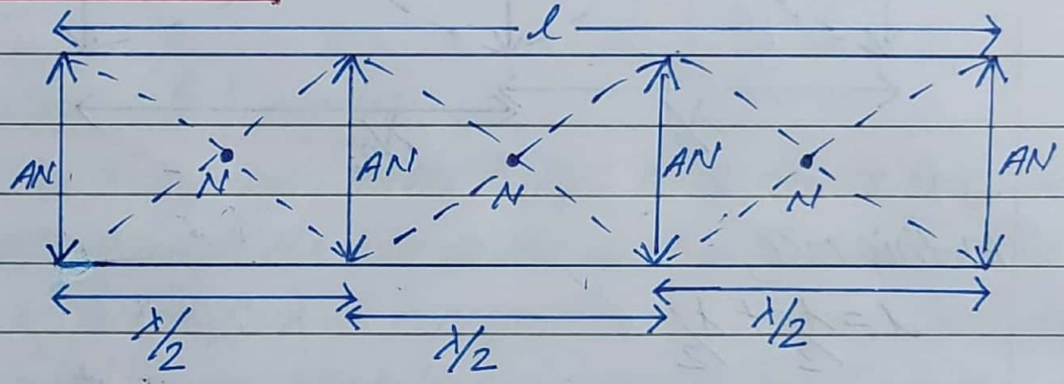
$$\Rightarrow f = \frac{2v}{2l}$$

$$\Rightarrow f = 2\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_2 = 2f_1}$$

This gives the second harmonic of the open organ pipe.

iii. Second overtone :-



In this case,

$$l = \lambda/2 + \lambda/2 + \lambda/2$$

$$\Rightarrow l = \frac{3\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{3}$$

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Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{3}\right)}$$

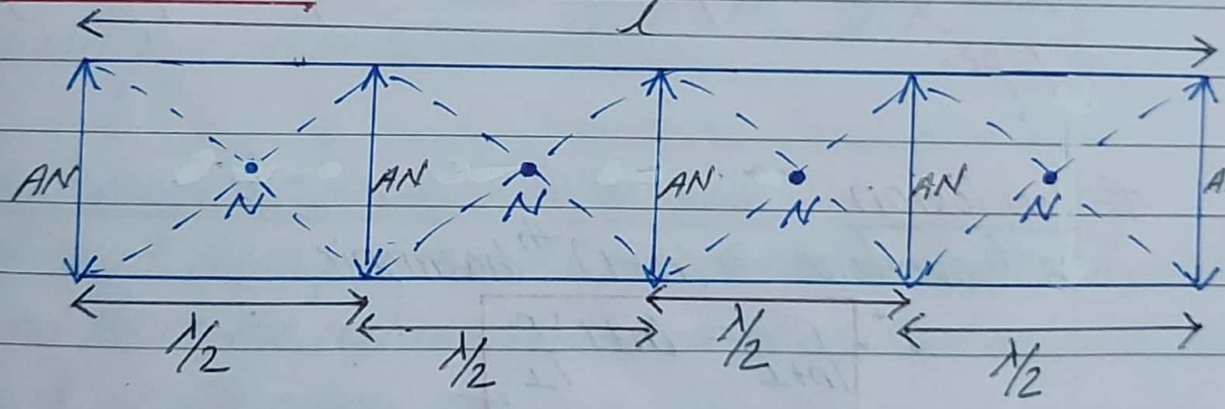
$$\Rightarrow f = \frac{3v}{2l}$$

$$\Rightarrow f = 3\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_3 = 3f_1}$$

This gives the third harmonic of the open organ pipe.

iv. Third overtone:-



In this case,

$$l = \lambda/2 + \lambda/2 + \lambda/2 + \lambda/2$$

$$\Rightarrow l = \frac{4\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2l}{4}$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\Rightarrow f = \frac{v}{\left(\frac{2l}{4}\right)}$$

$$\Rightarrow f = \frac{4v}{2l}$$

$$\Rightarrow f = 4\left(\frac{v}{2l}\right)$$

$$\Rightarrow \boxed{f_4 = 4f_1}$$

This gives the fourth harmonic of the open organ pipe.

Conclusion

i) n^{th} overtone $\Rightarrow (n+1)^{th}$ harmonic

$$\therefore \boxed{f_{n+1} = (n+1)f_1}$$

This gives the $(n+1)^{th}$ harmonic of the open organ pipe.

ii. Both odd as well as even harmonics are present. So, the quality of sound is rich. Hence, open organ pipes can be used as musical instruments.

End correction in organ pipes

The antinode is not formed exactly at the open end of an organ pipe, rather it is formed a little outside the open end of the organ pipe. The distance between the open end of the organ pipe and the antinode is called the end correction of that organ pipe. It is represented by 'c'.

According to Rayleigh

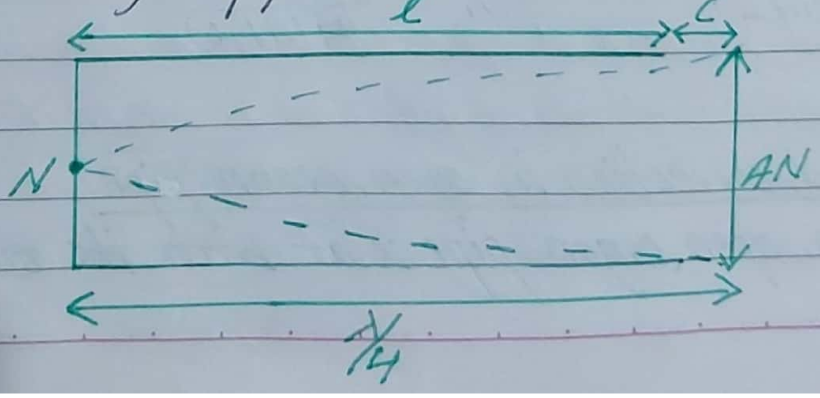
$$c = 0.3d$$

where, d is the diameter of the organ pipe.

With the correction, the actual frequency produced by the organ pipe can be determined.

1. End correction in closed organ pipe

A closed organ pipe is closed at one end and open at another end. Hence, the end correction is to be done at only one open end of the organ pipe.



In this case,

$$l + c = \frac{\lambda}{4}$$

$$\Rightarrow \lambda = 4(l+c)$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\boxed{\Rightarrow f_1 = \frac{v}{4(l+c)}}$$

This gives the fundamental frequency of closed organ pipe with end correction.

Thus,

$$\Rightarrow f_3 = 3f_1 = \frac{3v}{4(l+c)}$$

$$\Rightarrow f_5 = 5f_1 = \frac{5v}{4(l+c)}$$

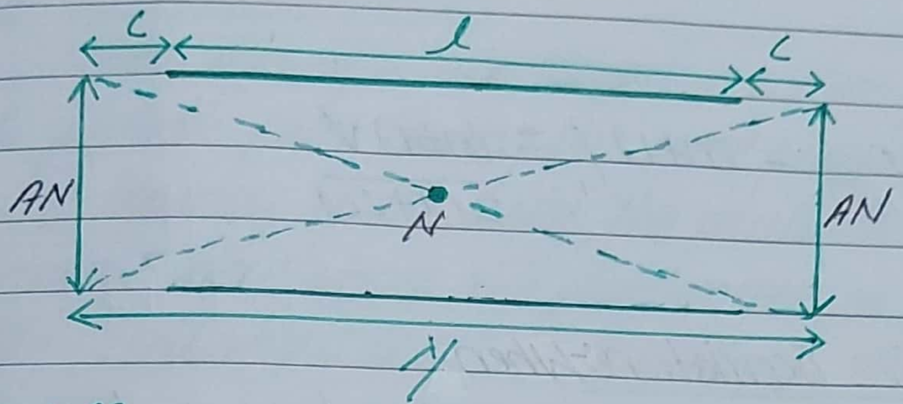
$$\Rightarrow f_7 = 7f_1 = \frac{7v}{4(l+c)}$$

$$\vdots$$

$$\Rightarrow f_{2n+1} = (2n+1) f_1 = \frac{(2n+1)v}{4(l+c)}$$

End correction in open organ pipe

An open organ pipe is at both its ends. Hence, the correction is to be done at both the open ends of the organ pipe.



In this case,

$$c + l + c = \lambda/2$$

$$\Rightarrow l + 2c = \frac{\lambda}{2}$$

$$\Rightarrow \lambda = 2(l + 2c)$$

Thus,

$$f = \frac{v}{\lambda}$$

$$\boxed{f_1 = \frac{v}{2(l+2c)}}$$

This gives the fundamental frequency of open pipe with end correction.

Thus,

$$\Rightarrow f_2 = 2f_1 = \frac{2v}{2(l+2c)}$$

$$\Rightarrow f_3 = 3f_1 = \frac{3v}{2(l+2c)}$$

$$\Rightarrow f_4 = 4f_1 = \frac{4v}{2(l+2e)}$$

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$$\Rightarrow f_{n+1} = (n+1)f_1 = \frac{(n+1)v}{2(l+2e)}$$

Free Oscillations:-

When a body is displaced from its equilibrium position and then free, it begins to oscillate with a definite amplitude and frequency. Such oscillation in the absence of any external resistance (eg friction, air resistance etc) is called free oscillation. The frequency of vibration of free oscillations is called its natural frequency.

Forced oscillations:-

When a body is maintained in a state of oscillation by external periodic force of frequency other than natural frequency of the body, the oscillation is called forced oscillation. In such oscillation, the frequency of oscillation is equal to the frequency of periodic force.

The external applied force on the body is called the driver and the body set into oscillations is called driven oscillation.

Resonant Oscillations:-

When a body is maintained in a state of oscillations by a periodic force having the same frequency of the natural frequency of the body, the oscillations are called resonant oscillations. The phenomenon of producing resonant oscillations is called resonance.

Resonance is a particular case of forced oscillations in which the two frequencies are equal.

Example:- When soldiers are passing through a bridge such that their frequency of marching and natural frequency of the bridge are equal, the bridge oscillates with maximum amplitude and hence may be destroyed. Due to this reason, soldiers are observed to break their steps while crossing a bridge.

Determination of velocity of sound in dry air at NTP using resonance tube apparatus.

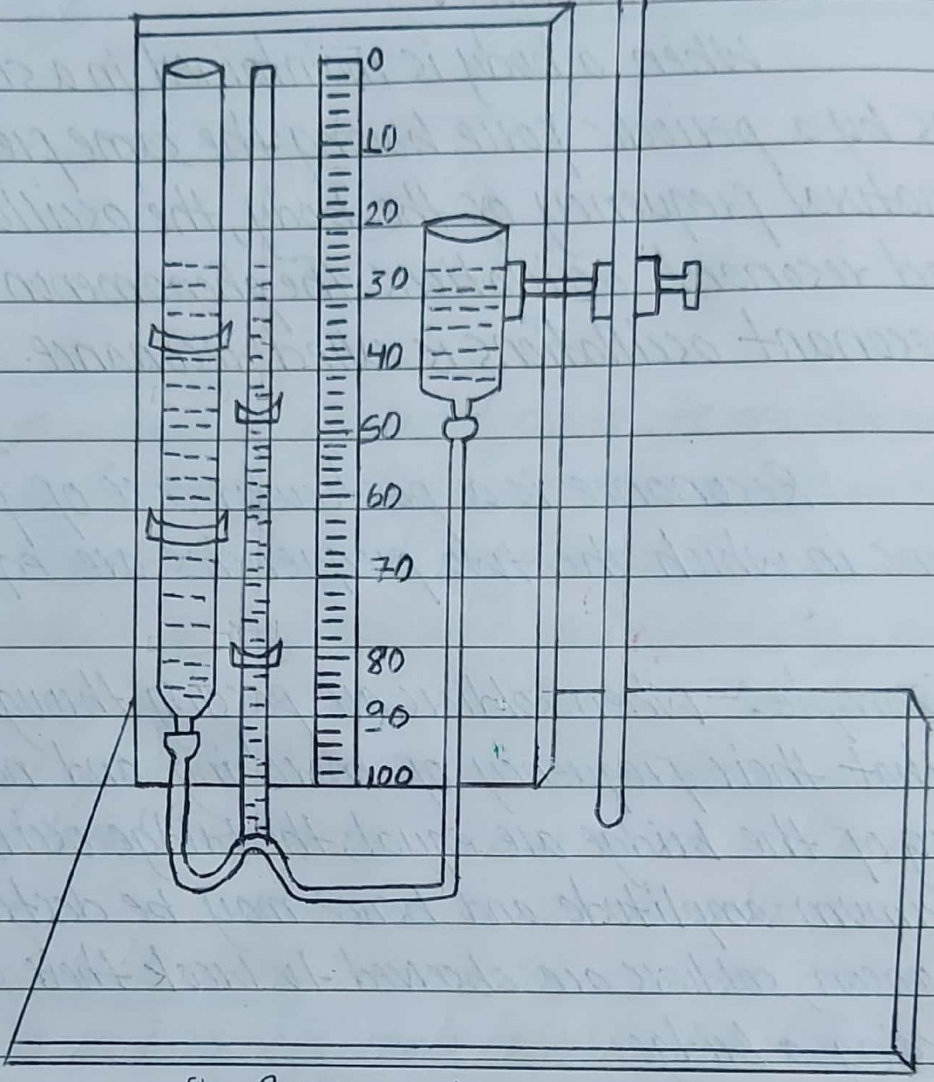
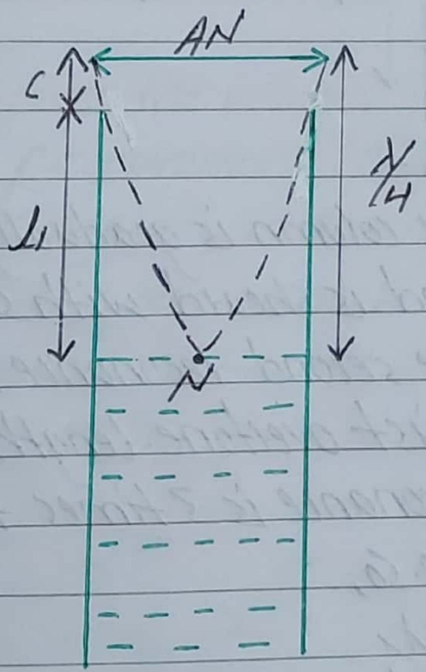


fig. Resonance tube apparatus

The apparatus consists of a one meter long glass tube of diameter 4cm. It is fitted on a vertical board with a meter scale attached to it as shown in the figure above. The tube is connected at its lower end by a rubber tube to water reservior which can be slided up and down. The glass tube and a part of the reservoir is filled with water. A fork is set into vibration and held horizontally above the mouth of the tube where the vibrating prongs of the fork force the air in the tube to vibrate. As the vibration is forced, the intensity of sound heard is small. The water level is adjusted to a position where a loud sound is heard. At this condition, the frequency of the tuning fork is same as the fundamental frequency of the pipe and the air inside pipe is set into resonance by the periodic force.

i) First Resonance

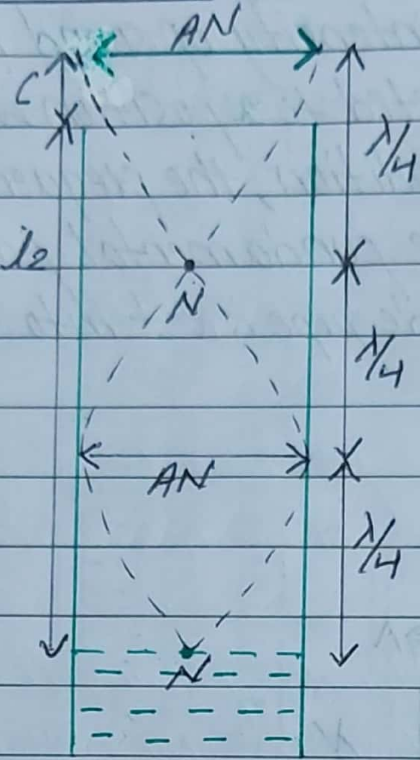


Here, the length of air column in the tube is denoted by l_1. Let C be the end correction and the fundamental mode of vibration is,

$$l_1 + C = \lambda/4 \quad \text{——— (i)}$$

$$Or,\ \lambda = 4(l_1 + c) \quad \text{—(i)}$$

ii) Second Resonance



Now, the length of air column is gradually increased till another loud sound is heard with same tuning fork. This is called the second resonance and this corresponds to the first overtone. Length l_2 of air column after this resonance is 3 times the length of the first resonance. So,

$$l_2 + c = \frac{3\lambda}{4}$$

$$Or,\ 3\lambda = 4(l_2 + c) \quad \text{—(ii)}$$

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Substracting eqⁿ (i) from (ii), we have

$$3\lambda - \lambda = 4(l_2 + c) - 4(l_1 + c)$$
$$\Rightarrow 2\lambda = 4l_2 + 4c - 4l_1 - 4c$$
$$\Rightarrow 2\lambda = 4(l_2 - l_1)$$
$$\Rightarrow \lambda = 2(l_2 - l_1)$$

Thus,

$$v = f\lambda$$
$$\Rightarrow v = f \cdot 2(l_2 - l_1)$$
$$\Rightarrow \boxed{v = 2f(l_2 - l_1)}$$

This gives the velocity of sound at lab conditions.

Here; f = frequency of tuning fork
l_1 = first resonating length
l_2 = second resonating length

We know that,

$$v \propto \sqrt{T}$$
$$\Rightarrow \frac{V_0}{v} = \sqrt{\frac{T_0}{T}}$$
$$\Rightarrow \frac{V_0}{v} = \sqrt{\frac{273}{t+273}}$$
$$\Rightarrow \frac{V_0}{v} = \sqrt{\frac{1}{\left(\frac{t+273}{273}\right)}}$$

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$$\Rightarrow \frac{V_0}{V} = \sqrt{\frac{1}{\frac{t}{273} + \frac{273}{273}}}$$

$$\Rightarrow \frac{V_0}{V} = \sqrt{\frac{1}{\left(1 + \frac{t}{273}\right)}}$$

$$\Rightarrow \frac{V_0}{V} = \sqrt{\left(1 + \frac{t}{273}\right)^{-1}}$$

$$\Rightarrow \frac{V_0}{V} = \left\{\left(1 + \frac{t}{273}\right)^{-1}\right\}^{1/2}$$

$$\Rightarrow \frac{V_0}{V} = \left(1 + \frac{t}{273}\right)^{-1/2}$$

$$\Rightarrow \frac{V_0}{V} = \left(1 + \frac{1}{273} \cdot t\right)^{-1/2}$$

$$\Rightarrow \frac{V_0}{V} = (1 + \alpha t)^{-1/2}$$

where, $\frac{1}{273}\ {}^\circ C^{-1} \Rightarrow$ temp^t constant

$$\Rightarrow \frac{V_0}{V} = \left(1 - \tfrac{1}{2}\alpha t\right)$$

$$\Rightarrow V_0 = v\left(1 - \frac{1}{2}\alpha t\right)$$

$$\Rightarrow \boxed{V_0 = 2f(l_2 - l_1)\left(1 - \frac{1}{2}\alpha t\right)}$$

Removing the effect of humidity,

$$\boxed{V_0 = 2f(l_2 - l_1)\left(1 - \frac{1}{2}\alpha t\right)\sqrt{\frac{P - 0.375 f_{aq}}{P}}}$$

Here,

f_{aq} = aqueous tension at lab condition.

This gives the velocity of sound in dry air at NTP.

From (ii), we have

$$3\lambda = 4(l_2 + c)$$

$$\Rightarrow 3 \cdot 4(l_1 + c) = 4(l_2 + c)$$

$$\Rightarrow 3l_1 + 3c = l_2 + c$$

$$\Rightarrow 3c - c = l_2 - 3l_1$$

$$\therefore \boxed{c = \frac{l_2 - 3l_1}{2}}$$

This gives the end correction of the resonance tube.